HYPOCOMPUTATION?

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ABSTRACT. Most research into hypercomputation focuses only on machines able to prove stronger results the basic Turing Machine, hence the phrase *hyper*computation. However, developing hypercomputational theories requires a deep understanding of computational theories: particularly the boundary between formal and informal theory present in all known computational theories. In this paper we argue for an investigation into computation from the informal side of this boundary, through the investigation of mathematical machines just below the boundary of computation. We will call this class of machines utilising a weaker definition of 'computation' *hypo*computational machines, and argue that by studying the assumptions used by these machine we should gain insights into the actions of machines such as the Turing Machine lying on the computational boundary. This in turn should provide a much firmer foundation for developing the theories required for hypercomputation, and may help in effort to build a hypercomputer.

1. INTRODUCTION

1.1. **Physical Hypercomputation.** Both popular, and more serious, discussions of hypercomputation often raise the question: "*can we actually* build *a hypercomputer?*" [**Cle2002**, **Cop2002**, **Sta2003**, **TS2002**]. But do we really have a deep enough understanding of *computation* to have a sensible discussion about the physical reality (or otherwise) of *hyper*computational machines?

In this paper we argue for a return to the foundations of computation, and specifically a return to the problems and discussions leading up to the formulation of Alan Turing's famous machines. It could be argued that a re-evaluation of computation has already been started by Carol E. Cleland [**Cle2004**]. However, while we would agree with Cleland in rejecting the excesses of the formalist program in mathematics, we do not believe computational theory should *necessarily* have any explicit physical realisation. It may be, though, that in broadening our mathematical understanding of computation, we may gain additional insight into machines that may legitimately be characterised as hypercomputational.

1.2. **The Boundary of Formal Theory.** In the history of mathematics, Alan Turing's description of computation arose in response to an earlier program by David Hilbert to systematise mathematical theory [**Bos1995**, **Gan1988**]. The 'failure' of Hilbert's program is well-known, and much discussed in the literature, so we will not go into the ultimate fate of his program here [**Bro1976**, **Det1986**, **Hil1967a**, **Wai1982**]. Yet, despite the demise of Hilbert's program, we know now little more than we did in the 1930s about the *nature* of that failure. Why, for instance, is formal theory a special case of informal mathematical theory? Could we transform all informal theories into formal theories by a better selection of formal theory, or could an informal theory somehow be fixed beyond the boundary of all formal theories? And what, exactly, is the nature of the boundary between formal and informal theory?

We do, however, know the computational theories developed by Alonzo Church, Alan Turing and Emil Post lie at the very edge of formal theory. Hence by studying computational theory we should be able to gain significant insight into the relationship between formal and informal mathematical theory [**Kle1971**, §60]. Would should then have a much better understanding of hypercomputational theories, through deepening our insight into the meaning of 'computation', 'formal theory' and 'in formal theory'.

2. A Return to Metamathematics

One of the best vantage points to study the boundary between formal and informal theory is the last legacy of Hilbert's program: metamathematics. Yet since Alfred Tarski's denunciation of metamathematics in the 1950s and '60s, metamathematics has essentially died as a serious area of study for most mathematicians [**Sin2001**].

Metamathematics was devised by David Hilbert to get around the many paradoxes in number theory and logic at the turn of the 20^{th} century. Prior to the 20^{th} century, logicians proved theoretical results *indirectly* by reference to previous results. For instance, a proof of consistency for the logical system **A** could be given by relating results in **A** to another system, say **B**, already known to be consistent. Hilbert, though, proposed scrapping these chains of inference, and instead asked for *direct* proof [**Hil1967b**, **Hil1976**]. In essence [**Kle1971**, §14, p. 55] (authors emphasis):

> . . . to prove the consistency of a theory directly, one should prove a proposition about the theory itself, i.e. specifically about all possible proofs of theorems in the theory. The mathematical theory whose consistency it is hoped to prove then becomes itself the *object* of a mathematical study, which Hilbert calls "metamathematics" or "proof theory".

In principle, the study of mathematical theories could be carried out using formal theories, such as Tarski's model theory. By using a formal theory in this way, we have no need to separate 'metamathematics' from 'mathematics' — both theories become both the object and the method of study.

However, before Tarski's work most mathematicians characterised the metatheory in which formal systems were described and studied as informal [**Kle1971**, §15]. And by using an informal metamathematics, we create a distinction between 'metamathematics' and 'mathematics'; since we have no need to ban methods from metamathematical theories just because we would not accept them in formal mathematical theories.

3. Informal Computation

3.1. **Defining Informal and Formal Theories.** In many recent discussions, though, the term 'informal' is more often used in the sense of "*vague*" or "*ill-defined*". To avoid confusion, and to better illustrate the value of an informal metatheory, we will briefly review the nature of both formal and informal mathematical theories.

The modern sense of 'formal' is due to David Hilbert, whereby a theory is only considered a *formal theory* if we can completely separate the form of a theory from its meaning (or interpretation) [**Hil1967b**, **Hil1976**]. As we saw from the quotation by Stephen Kleene above, Hilbert originally envisaged these questions over the meaning and interpretation of formal theory as being part of the metatheory (and the subject of metamathematics). In these metatheories, we have no need to completely isolate the form of a theory from its interpretation. If we choose not to separate the form of a theory from its interpretation, the theory is regarded as an *informal theory* — but this classification says nothing about the worth or utility of the theory.

3.2. **Using Formal and Informal Theory.** As a brief example, consider the well-known axioms of arithmetic proposed by Giuseppe Peano [**Kle1971**, p. 20][1]:

Clause 3.1. *0 is a* natural number*.*

Clause 3.2. *If n is a* natural number*, then n' is a* natural number*.*

Clause 3.3. *The only* natural numbers *are those given by Clause 3.1 and Clause 3.2.*

Clause 3.4. *For any natural numbers m and n, $m' = n'$ only if $m = n$.*

Clause 3.5. *For any natural number n, $n' \neq 0$.*

These axioms say nothing about many of the details we require for an interpretation of arithmetic. For instance, these axioms say nothing (and make no claims) about the *nature* of natural numbers: only how they form the sequence of *natural numbers*. Hence Peano's axioms are regarded as a formal theory in the sense defined above.

Moreover, we may derive other properties of arithmetic as defined by these axioms; again without any consideration as to the interpretation or the nature of the arithmetic we are considering. For instance, one of the most fundamental structural criteria we can derive from these axioms is the notion of *distinctness* between the natural numbers. For example, $0''''$ may be shown to be distinct from $0''$ (i.e. $0'''' \neq 0''$) as follows. By Clause 3.4 applied to $m = 0'''$ and $n = 0'$, $0'''' = 0''$ only if $0''' = 0'$. Again applying Clause 3.4 we get $0''' = 0'$ only if $0'' = 0$. But by Clause 3.5 with $n = 0'$, $0'' \neq 0$ and thus $0''''$ must be distinct from $0''$ [**Kle1971**, p. 20].

Having defined a concept in one formal theory, we may also seek related concepts in another. For instance, we may note formal logics require a similar notion of distinctness in the definition of a symbol. This observation naturally leads to the question of whether notions of 'distinctness' in these two formal theories are, in fact, related. Can we, for instance, relate 'distinctness' in a formalisation of arithematic to 'distinctness' in predicate calculus?

In modern mathematics, these questions are used addressed by creating a new formal theory in which both formal arithmatic and formal logic can be defined and related. However, this approach is not the only one. We could instead look to find a common *interpretation* of this formalisation of arithmatic and a formal logic. This common interpretative theory would be an informal mathematical theory, since we would have to look beyond the mere form of the theory in order to apply it.

Interpretive, informal metamathematical theories in this sense do exist, and (at least before 1940) were well studied. Nor are these informal theories necessarily any less rigorous than formal mathematical theories. For instance, Kurt Gödels' informal critique of Alfred Whitehead and Bertrand Russell's *Principia Mathmatica* is still considered a classic: and still has much to say to those developing formal theories in logic and number theory [**Göd1967**, **CP2000**].

For the value of informal mathematical theories, is precisely this fusion of the interpretation with a critique of the formal theory itself. We could indeed use a formal theory to separate these discussions, and such formal theories are indeed valuable. However, by choosing to remain with the informal theory, we gain a much more flexible framework in which to critique the boundary of the formal theory itself.

[1]Note: This presentation of Peano's Axioms follows that of Stephen Kleene, and differs in style from the original presentation by Giuseppe Peano [**Pea1967**]. For a full discussion of the differences, the reader is referred to Kleene's *Introduction to Metamathematics*, §6 [**Kle1971**].

3.3. **The Value of Informal Theories.** For instance, let us for a moment abandon Clause 3.5 in Peano's axiomatisation of arithmatic presented above. This move would be absurd in any formal theory, since it would remove the very concept of distinctness we need to formulate the formal theory in the first place[2]. Could we, though, formulate an informal theory respecting Clause 3.1 to Clause 3.4, but substituting an interpretative requirement for Clause 3.5?

In a recent paper by the author, we argue that we can indeed find an informal mathematical theory using the scheme laid out above [**Lov2006**]. Moreover, if we retain our informal metamathematics, we can use known (informal) results to relate this informal number theory to a weaker form of (informal) computation.

The value of this approach lies in the ability to study the formal/informal boundary of computation from both sides. We are not then limited only to using the extensive developments in (formal) computability and formal language theory since the 1940s. Instead we could augment these formal insights with those gained from studying the same problems using the more general informal theories. This, in turn, should foster the development of hypercomputational formal theories, by better defining the underlying assumptions and conditions required by those theories.

4. Summary

Developing hypercomputational theory requires a deep understanding of the boundary between a formal theory and the underlying informal theory. At present, most of the research effort has focused firmly on developing the formal mathematical theories, to develop stronger notions of computation. However, this focus ignores the much wider class of weaker, informal hypocomputational machines possible in mathematical theory.

Arguably, given the usual programming and implementation techniques for digital machines, physical hypocomputational machines are already in wide use. Nonetheless, by studying the theory of this weaker class of machines just below the boundary of computation, we should gain additional insight into the nature of computation. This, in turn, might advance our knowledge of hypercomputational theory to the point where we can indeed focus on how we might actually build a hypercomputer.

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